

# A LOWER BOUND FOR THE SCALAR CURVATURE OF CERTAIN STEADY GRADIENT RICCI SOLITONS

BENNETT CHOW, PENG LU, AND BO YANG<sup>1</sup>

We have the following result regarding steady Ricci solitons. See [1], [2], [4], [5], and [6] for some earlier works on the qualitative aspects of steady Ricci solitons.

**Theorem 1.** *Let  $(\mathcal{M}^n, g, f)$  be a complete steady gradient Ricci soliton with  $R_{ij} = -\nabla_i \nabla_j f$  and  $R + |\nabla f|^2 = 1$ . If  $\lim_{x \rightarrow \infty} f(x) = -\infty$  and  $f \leq 0$ , then  $R \geq \frac{1}{\sqrt{\frac{n}{2}+2}} e^f$ .*

*Proof.* Define  $\Delta_f = \Delta - \nabla f \cdot \nabla$ . Then  $\Delta_f f = -1$ ,  $\Delta_f R = -2|\text{Rc}|^2 \leq -\frac{2}{n}R^2$ , and  $\Delta_f(e^f) = -R e^f$ . For  $c \in \mathbb{R}$ ,

$$\Delta_f(R - ce^f) \leq -\frac{2}{n}R^2 + cR e^f \leq \frac{nc^2}{8}e^{2f}.$$

Using  $\Delta_f(e^{2f}) = 2e^{2f}(1 - 2R)$ , we compute for  $b \in \mathbb{R}$  that

$$(1) \quad \Delta_f(R - ce^f - be^{2f}) \leq \left( \frac{nc^2}{8} - 2b + 4bR \right) e^{2f}.$$

Suppose  $R - ce^f - be^{2f}$  is negative somewhere. Then, since  $R \geq 0$  by [3] and  $\lim_{x \rightarrow \infty} e^{f(x)} = 0$  by hypothesis, a negative minimum of  $R - ce^f - be^{2f}$  is attained at some point. By (1) and the maximum principle, at such a point we have

$$0 \leq \frac{nc^2}{8} - 2b + 4bR < \frac{nc^2}{8} - 2b + 4b(c + b)$$

since  $f \leq 0$ . Given  $c \in (0, \frac{1}{2}]$ , the minimizing choice  $b = \frac{1-2c}{4}$  yields  $\frac{(1-2c)^2}{4} < \frac{nc^2}{8}$ . We obtain a contradiction by choosing  $c = \frac{1}{\sqrt{\frac{n}{2}+2}}$ .  $\square$

**Remark.** *Given  $O \in \mathcal{M}$ , since  $|\nabla f| \leq 1$ , we have  $f(x) \geq f(O) - d(x, O)$  on  $\mathcal{M}$ . For the cigar soliton  $(\mathbb{R}^2, \frac{4(dx^2+dy^2)}{1+x^2+y^2})$  we have  $R = e^f$  assuming  $\max_{x \in \mathbb{R}^2} f(x) = 0$ . See [7] for an estimate for the potential functions of steady gradient Ricci solitons.*

## REFERENCES

- [1] Brendle, S. *Uniqueness of gradient Ricci solitons*. arXiv:1010.3684.
- [2] Cao, H.-D.; Chen, Q. *On locally conformally flat gradient steady Ricci solitons*. Trans. AMS, to appear.
- [3] Chen, B.-L. *Strong uniqueness of the Ricci flow*. J. of Diff. Geom. **82** (2009), 363–382.
- [4] Guo, H. *Area growth rate of the level surface of the potential function on the 3-dimensional steady Ricci soliton*. Proc. AMS **137** (2009), 2093–2097.
- [5] Hamilton, R. S. *The formation of singularities in the Ricci flow*. Surv. Diff. Geom. Vol. II, 7–136, Intern. Press, Cambridge, MA, 1995.
- [6] Munteanu, O.; Sesum, N. *On gradient Ricci solitons*. arXiv:0910.1105.
- [7] Wu, P. *Remarks on gradient steady Ricci solitons*. arXiv:1102.3018.

<sup>1</sup>Addresses. Bennett Chow and Bo Yang: Math. Dept., UC San Diego; Peng Lu: Math. Dept., U of Oregon.